

Lecture-Notes for The Students of
B.Sc. (Mathematics Honours), Semester - II
Course: MATH-H-DC03 (Real Analysis)

MODULE - 1: THE REAL NUMBERS

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We introduce the theme in a little bit way how the real numbers are developed. To do so we start with:

Natural Numbers

The Set of Natural Numbers

The natural numbers are the counting numbers $1, 2, 3, 4, 5, \dots$. We denote the set of all natural numbers by \mathbb{N} . Thus, we have

$$\mathbb{N} := \{1, 2, 3, 4, 5, \dots\}.$$

Peano Axioms or Peano Postulates

The set \mathbb{N} has the following self-evident fundamental properties:

- 1 $1 \in \mathbb{N}$.
- 2 $n \in \mathbb{N} \implies$ its successor $n + 1 \in \mathbb{N}$.
- 3 1 is not the successor of any element in \mathbb{N} .
- 4 $n, m \in \mathbb{N}$ having the same successor $\implies n = m$.
- 5 $A \subset \mathbb{N}$ such that $1 \in A$, and $n + 1 \in A$ whenever $n \in A \implies A = \mathbb{N}$.

Remark. The Peano axiom 5 is the basis of the mathematical induction.

Principle of Mathematical Induction

For any $n \in \mathbb{N}$, let $P(n)$ be a mathematical statement or a mathematical proposition which, in general, may or may not be true. If

- 1 $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true, and
- 2 $P(n)$ is true for $n = m + 1$ whenever it is true for $n = m$, i.e., $P(m + 1)$ is true whenever $P(m)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Algebraic Structures on \mathbb{N}

The algebraic operations of addition '+' and multiplication ' \cdot ' are binary operations on the set \mathbb{N} , i.e., the maps

$$\text{Addition } (+) : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

$$(m, n) \longmapsto m + n,$$

$$\text{Multiplication } (\cdot) : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

$$(m, n) \longmapsto m \cdot n$$

are well-defined.

Algebraic Structures on \mathbb{N}

The algebraic operations of addition '+' and multiplication '.' on the set \mathbb{N} have the following algebraic properties:

- 1 **Associativity of Addition:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{N}$.
- 2 **Commutativity of Addition:** $a + b = b + a$ for all $a, b \in \mathbb{N}$.
- 3 **Associativity of Multiplication:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{N}$.
- 4 **Commutativity of Multiplication:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{N}$.
- 5 **Existence of Multiplicative Identity:** There exists an element $1 \in \mathbb{N}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{N}$.
- 6 **Distributivity of Multiplication over Addition:** $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{N}$.

The set \mathbb{N} equipped with the algebraic operation addition '+' forms a *commutative semi-group*, i.e., $(\mathbb{N}, +)$ is a *commutative semi-group*.

The set \mathbb{N} equipped with the algebraic operation multiplication '.' forms a *commutative monoid*, i.e., (\mathbb{N}, \cdot) is a *commutative monoid*.

Order Structure on \mathbb{N}

Let us define a binary relation ' \leq ' the set \mathbb{N} by

$$a \leq b \text{ if and only if either } a = b \text{ or } a \text{ is less than } b$$

for all $a, b \in \mathbb{N}$. Then the binary relation ' \leq ' on \mathbb{N} has the following properties:

- 1 **Reflexivity:** For all $a \in \mathbb{N}$, $a \leq a$.
- 2 **Anti-Symmetry:** For all $a, b \in \mathbb{N}$, $a \leq b$ and $b \leq a \implies a = b$.
- 3 **Transitivity:** For all $a, b, c \in \mathbb{N}$, $a \leq b$ and $b \leq c \implies a \leq c$.
- 4 **Connexity:** For all $a, b \in \mathbb{N}$, $a \leq b$ or $b \leq a$.

For the first three properties, the relation ' \leq ' is a *partial order* on \mathbb{N} , and (\mathbb{N}, \leq) is a *partially ordered set*.

For all the four properties, the relation ' \leq ' is a *total order* or a *linear order* on \mathbb{N} , and (\mathbb{N}, \leq) is a *totally ordered set* or a *linearly ordered set*.

The set \mathbb{N} is well-ordered, i.e., every non-empty subset of \mathbb{N} has a least element.

The algebraic operations of addition ‘+’ and multiplication ‘.’ are well-defined on the set \mathbb{N} . But if we introduce the algebraic operation of subtraction ‘-’ on \mathbb{N} , then it becomes inadequate as the difference $m - n$ of two natural numbers m and n may not be a natural number. For instance, if $m = 3$ and $n = 5$, then their difference $3 - 5$ is not a natural number. Therefore, the algebraic operations of subtraction ‘-’ is not defined on \mathbb{N} . To make it well-defined we need to enlarge the set \mathbb{N} by including the numbers of the form $-n$, $n \in \mathbb{N}$, called the negatives of natural numbers and the number 0. The natural numbers together with their negatives and the number 0 are called integers.

The Set of Integers

We denote the set of all integers by \mathbb{Z} . Thus, we have

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \dots\}.$$

Algebraic Structures on \mathbb{Z}

The algebraic operations of addition '+' and multiplication '.' defined on the set \mathbb{Z} have the following algebraic properties:

- 1 **Associativity of Addition:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.
- 2 **Commutativity of Addition:** $a + b = b + a$ for all $a, b \in \mathbb{Z}$.
- 3 **Existence of Additive Identity:** There exists an element $0 \in \mathbb{Z}$ such that $a + 0 = a = 0 + a$ for all $a \in \mathbb{Z}$.
- 4 **Existence of Additive Inverse:** For each $a \in \mathbb{Z}$, there exists an element $-a \in \mathbb{Z}$ such that $a + (-a) = 0 = (-a) + a$.
- 5 **Associativity of Multiplication:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Z}$.
- 6 **Commutativity of Multiplication:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$.
- 7 **Existence of Multiplicative Identity:** There exists an element $1 \in \mathbb{Z}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{Z}$.
- 8 **Distributivity of Multiplication over Addition:** $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{Z}$.

The set \mathbb{Z} equipped with the algebraic operation addition '+' forms a *commutative group*, i.e., $(\mathbb{Z}, +)$ is a *commutative group*.

The set \mathbb{Z} equipped with the algebraic operation multiplication '.' forms a *commutative monoid*, i.e., (\mathbb{Z}, \cdot) is a *commutative monoid*.

Thus, on the set \mathbb{Z} , the algebraic operation of subtraction '-' is defined as the inverse of the algebraic operation of addition '+'. The difference of integers is defined by

$$a - b := a + (-b)$$

for all $a, b \in \mathbb{Z}$.

Order Structure on \mathbb{Z}

As it is on \mathbb{N} , the relation ' \leq ' is a *total order* or a *linear order* on \mathbb{Z} . Therefore, (\mathbb{Z}, \leq) is a *totally ordered* or a *linearly ordered set*. Here is a particular case of the property of connexity, viz., for all $a \in \mathbb{Z}$, exactly one of $a < 0$, $a = 0$, or $a > 0$ is true. This property is called the *Trichotomy Property*.

Rational Numbers

Now, all the three algebraic operations of addition '+', subtraction '-' and multiplication '.' are well-defined on the set \mathbb{Z} . But if we introduce the algebraic operation of division ' \div ' on \mathbb{Z} , then it also becomes inadequate as the quotient $\frac{m}{n}$ of two integers m and n may not be an integer. For instance, if $m = 3$ and $n = 5$, then their quotient $\frac{m}{n}$ is not an integer. So, we need to enlarge the set \mathbb{Z} by including the numbers of the form $\frac{m}{n}$, $m, n \in \mathbb{Z}$. Then the enlarged set would be well enough to define division as an inverse operation of multiplication. But there arises a very much technical and logical error, viz., what if we take $m \in \mathbb{Z}$ and $n = 0$? Well, in this case we get the quantities, namely, $\frac{m}{0}$, $m \in \mathbb{Z}$ which are conventionally undefined.

Definition

The rational numbers are all those numbers which can be expressed in the form $\frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$. We denote the set of all rational numbers by \mathbb{Q} . Thus, we have

$$\mathbb{Q} := \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

Algebraic Structures on \mathbb{Q}

The algebraic operations of addition '+' and multiplication '.' defined on the set \mathbb{Q} have the following algebraic properties:

- 1 **Associativity of Addition:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Q}$.
- 2 **Commutativity of Addition:** $a + b = b + a$ for all $a, b \in \mathbb{Q}$.
- 3 **Existence of Additive Identity:** There exists an element $0 \in \mathbb{Q}$ such that $a + 0 = a = 0 + a$ for all $a \in \mathbb{Q}$.
- 4 **Existence of Additive Inverse:** For each $a \in \mathbb{Q}$, there exists an element $-a \in \mathbb{Q}$ such that $a + (-a) = 0 = (-a) + a$.
- 5 **Associativity of Multiplication:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Q}$.
- 6 **Commutativity of Multiplication:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Q}$.
- 7 **Existence of Multiplicative Identity:** There exists an element $1 \in \mathbb{Q}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{Q}$.
- 8 **Existence of Multiplicative Inverse of Non-Zero Rationals:** For each $a \in \mathbb{Q} \setminus \{0\}$, there exists an element $\frac{1}{a} \in \mathbb{Q}$ such that $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$.
- 9 **Distributivity of Multiplication over Addition:** $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{Q}$.

The set \mathbb{Q} equipped with the algebraic operations of addition '+' and multiplication '.' forms a *field*, i.e., $(\mathbb{Q}, +, \cdot)$ is a *field*.

Order Structure on \mathbb{Q}









