B.COM (PART-III)HONOURS IN ACCOUNTING & FINANCE

SUBJECT CODE:3CH1

SUBJECT: BUSINESS ECONOMICS AND QUANTITATIVE TECHNIQUES

TOPIC-Maximum and minimum values of a function, Increasing and decreasing functions(Convexity and Concavity of a Function)

Convexity and Concavity of a Function

Up to this point we were dealing with the first order derivatives $f'(x)$ or $\frac{dy}{dx}$ .since the first derivative of a function is also a function of x, it too should be differentiable with respect to x ,provided the conditions of differentiability are first satisfied. The method of obtaining the second and even higher order derivatives of a function introduces nothing new. Once the first order derivative is obtained by the suitable method already discussed, the second order derivative is obtained by further use of the rules, this time applied to the first derivative is considered as a function of x.the second order derivative of a function $f(x)$ denoted by

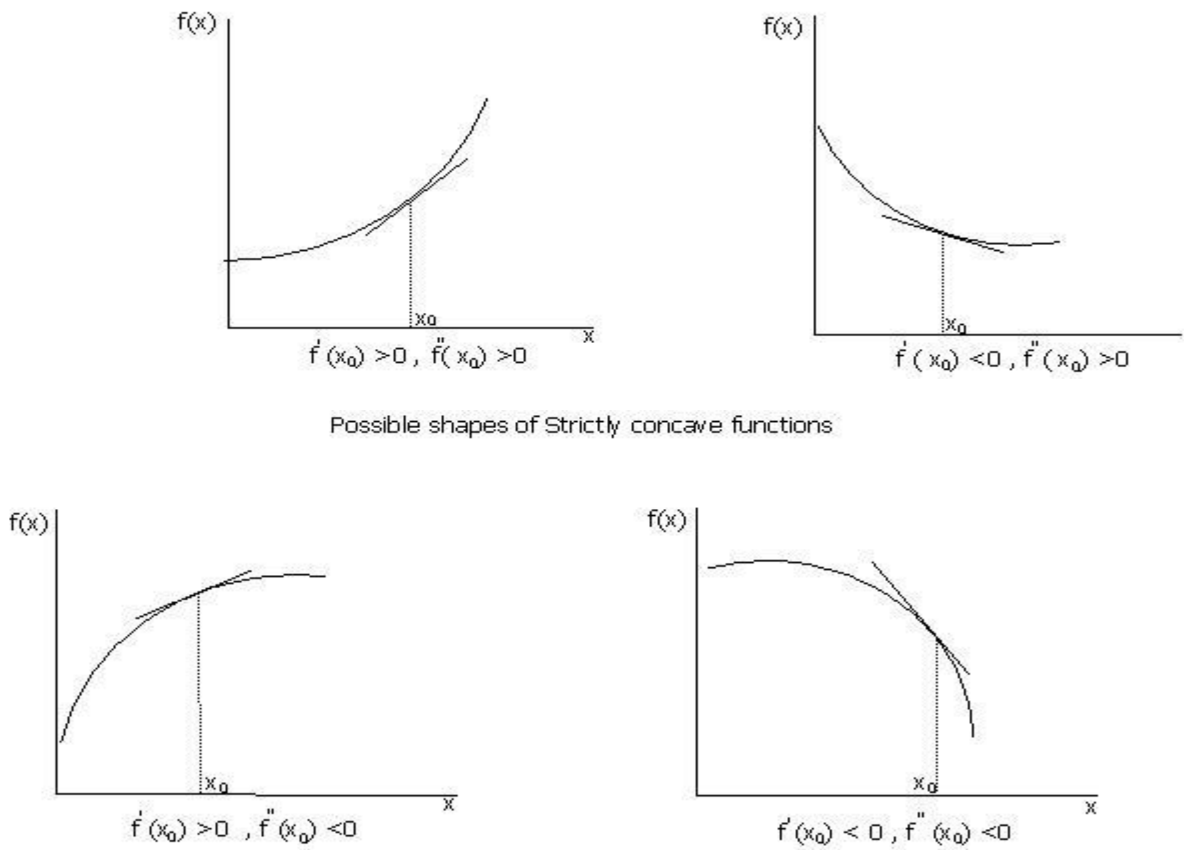
$$f''(x) \quad \text{or} \quad \frac{d(f'(x))}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2}$$

In economics we obtain many useful results by focussing on the first and second derivative of a function. The sign, positive or negative in particular, of a second derivative of a function leads us to an important and simple method of determining the convexity or concavity of a function.

- A twice differentiable function $f(x)$ is strictly convex at $x = x_0$ if $f''(x_0) > 0$. This means that the function $f(x)$ changes at an increasing rate as x increases through the value x_0.
- A twice differentiable function $f(x)$ is strictly concave at $x = x_0$ if $f''(x_0) < 0$. This means that the function $f(x)$ changes at a decreasing rate as x increases through the value x_0.

A function whose second derivative is sometimes positive and sometimes negative is neither convex nor concave everywhere. However, we can, sometimes find intervals over which the function is either convex or concave. Some possible shapes of strictly convex and concave functions are shown in fig. 2.6

FIG 2.6 possible shapes of the strictly convex functions



Example 2.11. check the convexity or concavity of the following.

a. $f(x) = 10 - x^2$ **b.** $f(x) = x^2$ **c.** $f(x) = -\frac{2}{3}(x)^3 + 10x^2 + 5x$; where $x \geq 0$

Solution.

a) Here $f(x) = 10 - x^2$

Differentiating with respect to x

$f'(x) = -2x$

Differentiating again with respect to x we have

$f''(x) = -2 < 0$

Since the second derivative is less than zero, it confirms that the given function is strictly concave.[7]

b) Given $f(x) = x^2$

Differentiating with respect to x

$f'(x) = 2x$

Differentiating again with respect to x we have

$f''(x) = 2 > 0$

Since the second derivative is greater than zero, it confirms that the given function is strictly convex.

c) Given $f(x) = -\frac{2}{3}(x)^3 + 10x^2 + 5x$

Differentiating with respect to x

$f'(x) = -2x^2 + 20x + 5$

Differentiating again with respect to x

$f''(x) = -4x + 20$

Since x can take values greater than or equal to zero. It follows that the function is convex on the interval [0, 5) and concave on the interval (5,∞).That is

$f''(x) = -4x + 20 > 0$ When $x < 5$ (convex)

$f''(x) = -4x + 20 < 0$ When $x > 5$ (concave)

Maximum and Minimum value of $y = f(x)$

A function $f(x)$ is said to have a maximum value at $x = x_0$, if $f(x_0)$ is greater than any other value that $f(x)$ can have in some suitably small neighbourhood of $x = x_0$. Alternatively a function $f(x)$ is said to have a maximum value at $x = x_0$, if $f(x_0)$ ceases to increase at $x = x_0$ and begins to decrease as x increases beyond x_0. similarly a function has a minimum at $x = x_0$ if it ceases to decrease at $x = x_0$ and begins to increase as x increases beyond x_0. the maximum and the minimum values together can be termed as the extreme values or stationary values of the function. It is pertinent to differentiate here between the relative or local maximum and the global or absolute maximum.

At global Maximum

$$f(x_0) \geq f(x) \qquad \text{for all } x$$

Whereas at relative maximum

$$f(x_0) \geq \quad f(x), \quad x_0 - \epsilon \ \leq x \leq \ x_0 + \epsilon \ \ x$$

Note that a global maximum must be a local one also, since if $f(x_0) \geq x$ for all x. Thus if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the global maximum. The global minimum of the function can be found analogously.

3.2 First derivative test for relative Maximum and minimum

A function $f(x)$ has a maximum value at $x = x_0$, if $f(x)$ ceases to increase at $x = x_0$ and begins to decrease as x increases beyond x_0

Thus, when x is slightly less than x_0 , $f'(x)$ is positive. When x is slightly greater than x_0, $f'(x)$ is negative.

$\therefore f'(x)$ Changes sign from positive to negative as x passes through the value x_0 .But it cannot change sign without passing through the value zero which must evidently be attained at $x = x_0$ itself.

Hence the following two conditions for a function $f(x)$ to have a maximum value at $x = x_0$ are:

- $f'(x) = 0$ at $x = x_0$
- $f'(x)$ Changes sign from positive to negative as x passes through the value x_0.

$f'(x) = 0$ at $x = x_0$ is referred to as the first order or the necessary condition for a function to be at relative extremum (maximum or minimum).the principle is even easier to see diagrammatically. In fig.2.7 (a), $x = x_0$ clearly gives a relative maximum of the function. At the point $(x_0, f(x_0))$, the tangent line to the function is horizontal which in other words mean the first derivative of the function is zero. If the necessary condition $f'(x_0) = 0$ is satisfied, then the change of derivative sign from positive to negative as x passes through the value x_0 serves as the **sufficient condition** for a function to be at relative maximum.

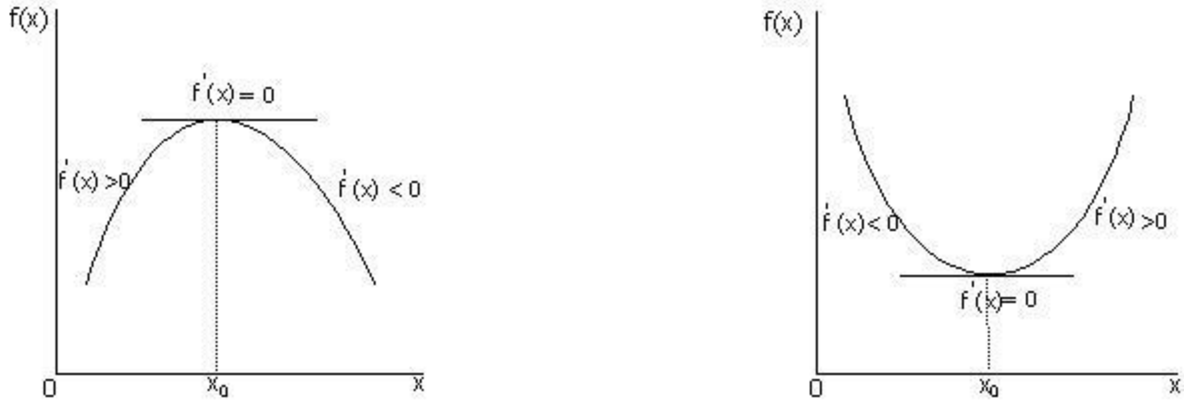
A function $f(x)$ has a minimum value at $x = x_0$ if it ceases to decrease at $x = x_0$ and begins to increase as x increases beyond x_0. Thus when x is slightly less than x_0, $f'(x)$ is negative and when x is slightly greater than x_0, $f'(x)$ is positive.

$\therefore f'(x)$ Changes sign from negative to positive as x passes through the value x_0 .But it cannot change sign without passing through the value zero which must evidently be attained at $x = x_0$ itself.

Hence the following two conditions for a function $f(x)$ to have a minimum value at $x = x_0$ are:

- $f'(x) = 0$ at $x = x_0$
- $f'(x)$ Changes sign from negative to positive as x passes through the value x_0.

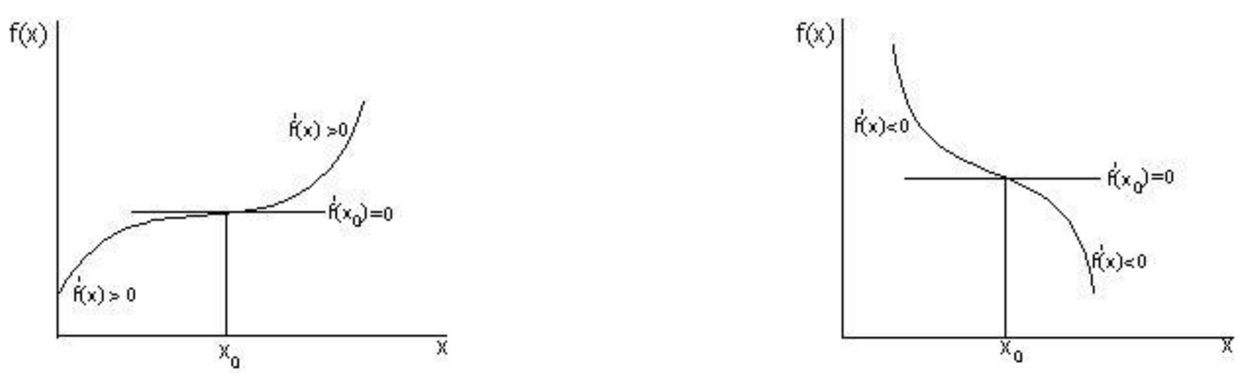
FIG. 2.7



(a) (b)

For the function $y = f(x)$ to have a maximum or minimum value at $x = x_0$, $f'(x) = 0$ at $x = x_0$. But if $f'(x) = 0$ at $x = x_0$, it is not essential that the function may have a maximum or a minimum value at $x = x_0$. It may happen that $f'(x)$ does not change sign from positive to negative or from negative to positive as x passes through the value x_0 and consequently, the function may go on increasing or decreasing. The possibility of such functions is depicted two parts of the fig. 2.8. In fig 2.8(a), the function is shown to have its first derivative zero at $x = x_0$, but the derivative does not change its sign from one side of $x = x_0$ to other. The derivative of the function remains positive on either side of the point x_0. thus according to the first derivative test point x_0 does not qualify for either maximum or minimum of a function. Similarly fig 2.8 (b) portray the case where on either sides of x_0, $f'(x)$ does not change sign but remains negative to the left and to the right of $x = x_0$. Both the cases in Fig. 2.8 exemplifies x_0 as an **inflection point**. An inflection point is a point on the graph where the function crosses its tangent line and changes from convex to concave or vice versa.

FIG.2.8



(a) (b)

Example 1. Examine the following functions for maximum and minimum values.

a. $f(x) = x^2 - 4x + 5$ **b.** $f(x) = \frac{(x-1)(x-6)}{x-10}$

Solution:

a. Given $f(x) = x^2 - 4x + 5$

Differentiating with respect to x, we have

$$f'(x) = 2x - 4$$

To get the critical value(s) we set the first derivative of the function equal to zero

$$2x - 4 = 0 \Rightarrow x = 2$$

Now to verify whether the function is at relative maximum or minimum, we will choose a point in the immediate neighbourhood of $x = 2$ and check the change in the sign of the first derivative. It is obvious, when $x < 2$, the sign of the first derivative is negative and when $x > 2$, the sign of the first derivative is positive. Since the first derivative changes sign from negative to positive, the function is at relative minimum at $x = 2$.

b. Given $f(x) = \frac{(x-1)(x-6)}{x-10}$

Differentiating with respect to x

$$f'(x) = \frac{(x-10)(2x-7) - (x^2 - 7x + 6).1}{(x-10)^2}$$

To get the critical values, set the first derivative equal to zero

$$\frac{(x-10)(2x-7) - (x^2 - 7x + 6).1}{(x-10)^2} = 0 \Rightarrow x = 4 \text{ or } x = 16$$

Applying first derivative test one by one

(i) When x is slightly < 4 , $f'(x) =$ positive

When x is slightly >4, $f'(x) =$ negative

Since the first derivative changes sign from positive to negative, this confirms that the function is at relative maximum at $x = 4$

(ii) When x is slightly <16, $f'(x)$ = negative

When x is slightly >16, $f'(x) = \text{positive}$

Since the first derivative changes sign from negative to positive, the function is at relative minimum at $x = 16$.

3.3 Second Derivative Test

We saw up till that $f'(x_0)$ =0 does not in itself tell us whether x_0 yields a maximum or minimum of a function, it was the sign of the derivative after fulfilling the necessary condition that enables us to determine the maximum or minimum of a function.

An alternative criterion for maximum or minimum of function can be given in terms of second derivative. This test is more convenient to use than the first- derivative test, because it does not require us to check the derivative sign to both left and right of point x_0. Assuming that the function is twice differentiable, if the value of the first derivative of a function at x= x_0 is zero i.e. $f'(x_0) = 0$, then

(i) The function will be at relative maximum at x= x_0 if $f''(x_0) < 0$
(ii) The function will be at relative minimum at x= x_0 if $f''(x_0) > 0$

It is obvious from above that the first condition is as before a necessary condition. The sign of the second derivative is a sufficient condition indicating situations in which only a maximum or a minimum value of a function can occur. This method however has a drawback that no concrete conclusion can be drawn in the event if $f''(x_0) = 0$.under such circumstances we can go back to the first derivative test to decide the relative extremum of the function.

Example 2. Find the relative maxima and minima of the following functions by the second derivative test.

a. $f(x) = 2x^2 - x^3$ **b.** $f(x) = 2{-}9x^2 + 6x - 10$ **c.** $f(Q) = Q^2 - 5Q + 8$

Solution:

a. Given $f(x) = 2x^2 - x^3$

To find the stationary values set first derivative equal to zero

$$f'(x) = 4x - 3x^2 = 0 \Rightarrow x = 0 \text{ or } x = \frac{4}{3}$$

For relative extremum we will find out the second derivative of the function and check the sign of the second derivative at the respective stationary values

$$f''(x) = 4 - 6x$$

When x =0, $f''(0) = 4 > 0$

Therefore the function is at minimum at x =0

When x = $\frac{4}{3}$, $f''\left(\frac{4}{3}\right) = -4 < 0$

Therefore the function is a maximum at x = $\frac{4}{3}$

b. Given $f(x) = 2-9x^2 + 6x - 10$

To find the stationary values set first derivative equal to zero

$$f'(x) = -18x + 6 = 0 \Rightarrow x = -\frac{1}{3}$$

For relative extremum we will find out the second derivative of the function and check the sign of the second derivative at the respective stationary value

$$f''(x) = -18$$

When $x = -\frac{1}{3}$, $f''\left(-\frac{1}{3}\right) = -18 < 0$

Therefore the function is a maximum at x = $-\frac{1}{3}$

c. Given $f(Q) = Q^2 - 5Q + 8$

To find the stationary values set first derivative equal to zero

$$f'(Q) = 2Q - 5 = 0 \Rightarrow Q = \frac{5}{2}$$

For relative extremum we will find out the second derivative of the function and check the sign of the second derivative at the respective stationary value

$$f''(Q) = 2$$

When $Q = \frac{5}{2}$, $f''\left(\frac{5}{2}\right) = 2 > 0$

Therefore the function is at minimum at $Q = \frac{5}{2}$

THANK YOU

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