Module - 1: The Real Numbers

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We introduce the theme in a little bit way how the real numbers are developed. To do so we start with:

Natural Numbers

The Set of Natural Numbers

The natural numbers are the counting numbers $1, 2, 3, 4, 5, \ldots$ We denote the set of all natural numbers by $\mathbb{N}$. Thus, we have

$$\mathbb{N} := \{1, 2, 3, 4, 5, \ldots\}.$$

Peano Axioms or Peano Postulates

The set $\mathbb{N}$ has the following self-evident fundamental properties:

1. $1 \in \mathbb{N}$.
2. $n \in \mathbb{N} \implies$ its successor $n + 1 \in \mathbb{N}$.
3. 1 is not the successor of any element in $\mathbb{N}$.
4. $n, m \in \mathbb{N}$ having the same successor $\implies n = m$.
5. $A \subset \mathbb{N}$ such that $1 \in A$, and $n + 1 \in A$ whenever $n \in A \implies A = \mathbb{N}$.

Remark. The Peano axiom 5 is the basis of the mathematical induction.

Principle of Mathematical Induction

For any $n \in \mathbb{N}$, let $P(n)$ be a mathematical statement or a mathematical proposition which, in general, may or may not be true. If

1. $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true, and
2. $P(n)$ is true for $n = m + 1$ whenever it is true for $n = m$, i.e., $P(m + 1)$ is true whenever $P(m)$ is true,

then $P(n)$ if true for all $n \in \mathbb{N}$.

Algebraic Structures on $\mathbb{N}$

The algebraic operations of addition '+' and multiplication '$\cdot$' are binary operations on the set $\mathbb{N}$, i.e., the maps

$$\begin{aligned} \text{Addition } (+) : \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (m, n) &\longmapsto m + n, \\ \text{Multiplication } (\cdot) : \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (m, n) &\longmapsto m \cdot n \end{aligned}$$

are well-defined.

Algebraic Structures on $\mathbb{N}$

The algebraic operations of addition '$+$' and multiplication '$\cdot$' on the set $\mathbb{N}$ have the following algebraic properties:

1. Associativity of Addition: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{N}$.
2. Commutativity of Addition: $a + b = b + a$ for all $a, b \in \mathbb{N}$.
3. Associativity of Multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{N}$.
4. Commutativity of Multiplication: $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{N}$.
5. Existence of Multiplicative Identity: There exists an element $1 \in \mathbb{N}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{N}$.
6. Distributivity of Multiplication over Addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{N}$.

The set $\mathbb{N}$ equipped with the algebraic operation addition '$+$' forms a *commutative semi-group*, i.e., $(\mathbb{N}, +)$ is a *commutative semi-group*.

The set $\mathbb{N}$ equipped with the algebraic operation multiplication '$\cdot$' forms a *commutative monoid*, i.e., $(\mathbb{N}, \cdot)$ is a *commutative monoid*.

Order Structure on $\mathbb{N}$

Let us define a binary relation '$\leq$' the set $\mathbb{N}$ by

$$a \leq b \text{ if and only if either } a = b \text{ or } a \text{ is less than } b$$

for all $a, b \in \mathbb{N}$. Then the binary relation '$\leq$' on $\mathbb{N}$ has the following properties:

1. Reflexivity: For all $a \in \mathbb{N}$, $a \leq a$.
2. Anti-Symmetry: For all $a, b \in \mathbb{N}$, $a \leq b$ and $b \leq a \implies a = b$.
3. Transitivity: For all $a, b, c \in \mathbb{N}$, $a \leq b$ and $b \leq c \implies a \leq c$.
4. Connexity: For all $a, b \in \mathbb{N}$, $a \leq b$ or $b \leq a$.

For the first three properties, the relation '$\leq$' is a *partial order* on $\mathbb{N}$, and $(\mathbb{N}, \leq)$ is a *partially ordered set*.

For all the four properties, the relation '$\leq$' is a *total order* or a *linear order* on $\mathbb{N}$, and $(\mathbb{N}, \leq)$ is a *totally ordered set* or a *linearly ordered set*.

The set $\mathbb{N}$ is well-ordered, i.e., every non-empty subset of $\mathbb{N}$ has a least element.

Integers

The algebraic operations of addition '+' and multiplication '$\cdot$' are well-defined on the set $\mathbb{N}$. But if we introduce the algebraic operation of subtraction '$-$' on $\mathbb{N}$, then it becomes inadequate as the difference $m - n$ of two natural numbers m and n may not be a natural number. For instance, if $m = 3$ and $n = 5$, then their difference $3 - 5$ is not a natural number. Therefore, the algebraic operations of subtraction '$-$' is not defined on $\mathbb{N}$. To make it well-defined we need to enlarge the set $\mathbb{N}$ by including the numbers of the form $-n$, $n \in \mathbb{N}$, called the negatives of natural numbers and the number 0. The natural numbers together with their negatives and the number 0 are called integers.

The Set of Integers

We denote the set of all integers by $\mathbb{Z}$. Thus, we have

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \dots\}.$$

Algebraic Structures on $\mathbb{Z}$

The algebraic operations of addition '+' and multiplication '·' defined on the set $\mathbb{Z}$ have the following algebraic properties:

1. Associativity of Addition: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.
2. Commutativity of Addition: $a + b = b + a$ for all $a, b \in \mathbb{Z}$.
3. Existence of Additive Identity: There exists an element $0 \in \mathbb{Z}$ such that $a + 0 = a = 0 + a$ for all $a \in \mathbb{Z}$.
4. Existence of Additive Inverse: For each $a \in \mathbb{Z}$, there exists an element $-a \in \mathbb{Z}$ such that $a + (-a) = 0 = (-a) + a$.
5. Associativity of Multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Z}$.
6. Commutativity of Multiplication: $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$.
7. Existence of Multiplicative Identity: There exists an element $1 \in \mathbb{Z}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{Z}$.
8. Distributivity of Multiplication over Addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{Z}$.

The set $\mathbb{Z}$ equipped with the algebraic operation addition '+' forms a *commutative group*, i.e., $(\mathbb{Z}, +)$ is a *commutative group*.

The set $\mathbb{Z}$ equipped with the algebraic operation multiplication '$\cdot$' forms a *commutative monoid*, i.e., $(\mathbb{Z}, \cdot)$ is a *commutative monoid*.

Thus, on the set $\mathbb{Z}$, the algebraic operation of subtraction '$-$' is defined as the inverse of the algebraic operation of addition '+'. The difference of integers is defined by

$$a - b := a + (-b)$$

for all $a, b \in \mathbb{Z}$.

Order Structure on $\mathbb{Z}$

As it is on $\mathbb{N}$, the relation '$\leq$' is a *total order* or a *linear order* on $\mathbb{Z}$. Therefore, $(\mathbb{Z}, \leq)$ is a *totally ordered* or a *linearly ordered set*. Here is a particular case of the property of connexity, viz., for all $a \in \mathbb{Z}$, exactly one of $a < 0$, $a = 0$, or $a > 0$ is true. This property is called the *Trichotomy Property*.

Rational Numbers

Now, all the three algebraic operations of addition '$+$', subtraction '$-$' and multiplication '$\cdot$' are well-defined on the set $\mathbb{Z}$. But if we introduce the algebraic operation of division '$\div$' on $\mathbb{Z}$, then it also becomes inadequate as the quotient $\frac{m}{n}$ of two integers m and n may not be an integer. For instance, if $m = 3$ and $n = 5$, then their quotient $\frac{m}{n}$ is not an integer. So, we need to enlarge the set $\mathbb{Z}$ by including the numbers of the form $\frac{m}{n}$, $m, n \in \mathbb{Z}$. Then the enlarged set would be well enough to define division as an inverse operation of multiplication. But there arises a very much technical and logical error, viz., what if we take $m \in \mathbb{Z}$ and $n = 0$? Well, in this case we get the quantities, namely, $\frac{m}{0}$, $m \in \mathbb{Z}$ which are conventionally undefined.

Definition

The rational numbers are all those numbers which can be expressed in the form $\frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$. We denote the set of all rational numbers by $\mathbb{Q}$. Thus, we have

$$\mathbb{Q} := \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

Algebraic Structures on $\mathbb{Q}$

The algebraic operations of addition '+' and multiplication '·' defined on the set $\mathbb{Q}$ have the following algebraic properties:

1. Associativity of Addition: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Q}$.
2. Commutativity of Addition: $a + b = b + a$ for all $a, b \in \mathbb{Q}$.
3. Existence of Additive Identity: There exists an element $0 \in \mathbb{Q}$ such that $a + 0 = a = 0 + a$ for all $a \in \mathbb{Q}$.
4. Existence of Additive Inverse: For each $a \in \mathbb{Q}$, there exists an element $-a \in \mathbb{Q}$ such that $a + (-a) = 0 = (-a) + a$.
5. Associativity of Multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Q}$.
6. Commutativity of Multiplication: $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Q}$.
7. Existence of Multiplicative Identity: There exists an element $1 \in \mathbb{Q}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{Q}$.
8. Existence of Multiplicative Inverse of Non-Zero Rationals: For each $a \in \mathbb{Q} \setminus \{0\}$, there exists an element $\frac{1}{a} \in \mathbb{Q}$ such that $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$.
9. Distributivity of Multiplication over Addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{Q}$.

The set $\mathbb{Q}$ equipped with the algebraic operations of addition '$+$' and multiplication '$\cdot$' forms a *field*, i.e., $(\mathbb{Q}, +, \cdot)$ is a *field*.

Order Structure on $\mathbb{Q}$